A New Method for Choosing the Regularization Parameter in the Transmembrane Potential Based Inverse Problem of ECG

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Abstract

In this paper we propose an iteratively regularized Gauss-Newton method to solve the inverse ECG problem and efficiently choose the parameter of regularization. The classical stopping criterium for this regularization technique - Morozov discrepancy principle, cannot be used in our application because the noise level estimate and problem model error are typically not available. We formulate the stopping rule based on the statistical formulation of the parameter and the physiological nature of the sought solution. With Laplace operator as a regularization matrix, the regularization parameter can be seen as an indirect measure of deviation in the solution: smaller parameters lead to a broader solution range. From our knowledge about electrophysiology of the heart we can assume values of -85 mV and +25 mV as a lower and an upper estimates for transmembrane potentials. Under this assumption we stop Gaus-Newton iteration as soon as the difference between solution smallest and largest values achieves 110 mV. Three simulation protocols confirm our ansatz: the proposed method was compared with the commonly used in the field L-curve based Tikhonov method, showing superior performance during an initial phase of an ectopic heart activation sequence.

1. Introduction

The inverse problem of ECG consists in reconstruction of electrophysiological heart activity from remote measurements on the thorax surface. This noninvasive procedure has a great potential for diagnosis and treatment planning of cardiac diseases and effective catheter ablation.

The mathematical basis underlying the ECG inverse problem can be described with a static bidomain heart model and monodomain thorax model introduced by Geselowitz and Miller [1].

There are several formulations of the problem, depending on the information to be recovered: potential based and activation times based approaches. Potential based problems assume endo-, epicardial or transmembrane potentials as source models. In this work we solve the problem in terms of transmembrane potentials in the heart volume.

With use of finite element discretization of a problem domain the linear relationship between N unknowns and M available measurements can be established:

$$Ax = y, \tag{1}$$

where A is a $M \times N$ matrix, x - a $N \times 1$ vector of unknown transmembrane potentials, y - a $M \times 1$ vector of body surface potential maps (BSPM). The matrix A is typically called leadfield because of its structure: the matrix columns correspond to the source poitns in the heart and characterize their contribution to the ECG signal [2].

Due to attenuation and smoothing of the electric fields in the body volume conductor the inverse problem of ECG is ill-posed, i.e. the solution is highly unstable with respect to uncertainties in the measured BSPM. In order to obtain a stable meaningful solution Tikhonov regularization is invoked [3]:

$$x = argmin\{\|Ax - y\|^2 + \lambda\|Lx\|^2\}, \tag{2}$$

with $\|\cdot\|$ being a norm in Euclidean metric. The first term in the expression represents the data misfit or residual error while the second term - regularizer imposes constraints on the desired solution. For the inverse problem of ECG three following choices for the regularization matrix are most common: zero-order Tikhonov takes L to an identity matrix thus penalizing the magnitude of the source signal; first-order Tikhonov uses gradient operator and imposes regularization on the spatial derivative of potentials; second-order Tikhonov requires smooth spatial curvature with Laplace operator as L matrix.

The regularization parameter can be interpreted as the trade-off between model (residual) error and solution desired properties. The determination of the optimal regularization parameter is itself a challeging task. All existing parameter choice methods can be classified according to the input they require:a priori methods, a posteriori methods, data-driven methods [4]. In the ECG inverse problem the noise and model error information are not available, which makes a posteriroi methods impracticle.

Among data-driven heuristic methods L-curve introduced by Hansen [5] received the most attention. It is based on the observation that log-log graph $(\|Ax_n - y\|, \|Lx_n\|)$ for a monotone set λ_n of regularization parameters often has a characteristic L shape.

2. Methods

Regularization techniques for ill-posed underdetermined problems have been a subject of intensive studies during last decades. From the nesessary condition for x to be a minimum of (2) the closed form solution for transmembrane potentials reads:

$$x = (A^T A + \lambda L^T L)^{-1} A^T y \tag{3}$$

The inversion in (3), depending on the problem size, can be efficiently done iteratively or with the use of singular value decomposition (SVD).

For the transmembrane potential based inverse problem of ECG a common choice for the regularization matrix L is Laplace operator - matrix representing second spatial derivate of the source signal. It damps high-frequency noise and delivers smooth approximation of the sought solution.

We obtain the regularization operator from finite element discretization of Laplace equation in the heart domain.

In standard Tikhonov approach the problem (3) is solved for a set of regularization parameters λ and then the optimal solution is picked acoording to the L-curve criterion.

2.1. Gauss-Newton optimization method

For an at least twice differentiable function $f(x)$ Taylor series expansion around point x_k up to the second order approximation is given by:

$$f(x_k + \Delta x) \approx f(x_k) + \Delta x f'(x_k) + \Delta x f''(x_k) \Delta x^T \tag{4}$$

with $x_{k+1} = x_k + \Delta x$. The minimum in (4) is achieved for Δx being a critical point:

$$f'(x_k) + f''(x_k)\Delta x = 0. \tag{5}$$

From this condition the descent Newton direction reads:

$$\Delta x = -f'(x_k)/f''(x_k) \tag{6}$$

which in discrete case can be written as:

$$\Delta x = -H^{-1} g, \tag{7}$$

where g and H are the gradient and Jacobian matrix of the function $f(x)$ at the point x_k respectively. Then the update formula for the solution x is:

$$x_{k+1} = x_k - H^{-1} g \tag{8}$$

For quadratic functions the algorithm converges in one iteration, because (4) is an exact equality in this case .

2.2. Iteratively regularized Gauss-Newton method

Iteratively regularized Gauss-Newton method can be considered as Tikhonov regularization with a variable regularization parameter [6]. For our problem the gradient and Hessian matrix of (2) can be easily computed:

$$g = A^T(Ax - y) + \lambda L^T L x \tag{9}$$

$$H = A^T A + \lambda L^T L. \tag{10}$$

This leads to the iterative solution of (2) with the update formula (8):

$$x_{k+1} = x_k - (A^T A + \lambda_k L^T L)^{-1} [A^T(Ax - y) + \lambda_k L^T L x] \tag{11}$$

where λ_k is a monotonically decreasing sequence satisfying

$$\lambda_k > 0, \quad 1 \leq \frac{\lambda_k}{\lambda_{k+1}} \leq c, \quad \lim_{k\to\infty} \lambda_k = 0 \tag{12}$$

The discrepancy principle is usually used as an a-posteriori stopping criterion for $k_* = k_*(\Delta)$:

$$\|Ax_* - y\| \leq \tau\Delta < \|Ax_k - y\|, 0 \leq k < k_* \tag{13}$$

where $\tau > 1$ and Δ is upper estimate for the problem error, $\|\delta\| \leq \Delta$. When the information about noise level is unavailable the L-curve or other methods are used to choose the optimal stopping index.

2.3. Selection of regularization parameter

If we consider the following maximum a posteriori problem (MAP) [7]:

$$Lx = v, \quad v \sim N(0, \lambda_x I) \tag{14}$$

$$y = Ax + w, \quad w \sim N(0, \lambda_w I) \tag{15}$$

the corresponding optimal MAP estimate for x would be:

$$x_{map} = argmin \|Ax - y\|^2 + \frac{\lambda_w}{\lambda_x} \|Lx\|^2. \tag{16}$$

This expression is identical to (2) with $\lambda = \frac{\lambda_w}{\lambda_x}$. From these considerations the regularization parameter can be treated as the ratio of the noise standard deviation in data to prior model (spatial distribuation in our case) standard deviation.

Although the noise in data is not available, it can be concluded that smaller regularization parameters allow for a broader solution distribution. This observation was also confirmed with experiments: for each following iteration (11) the difference between smallest and largest values in x_{k+1} is increased.

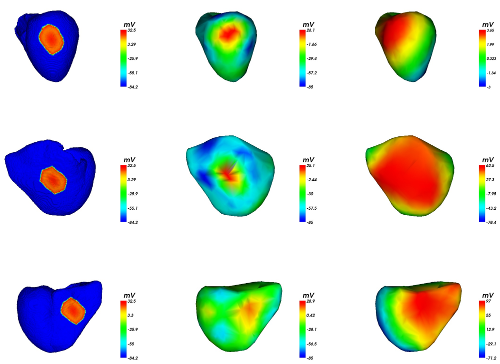

Figure 1. Transmembrane potentials for the time instance 30 ms with 30 dB noise input. From left to right: simulated data, reconstruction with the proposed algorithm, Tikhonov reconstruction with the regularization parameter from the L-curve method. From top to bottom: simulation1-3.

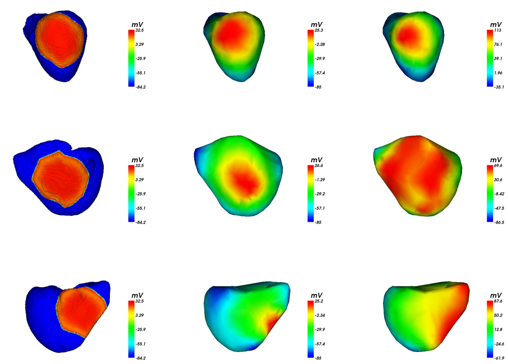

Figure 2. Transmembrane potentials for the time instance 60 ms with 30 dB noise input. From left to right: simulated data, reconstruction with the proposed algorithm, Tikhonov reconstruction with the regularization parameter from the L-curve method. From top to bottom: simulation1-3.

For the transmembrane potentials the physiological solution range is approximately known: the values of -85 mV for the resting (not depolarized) and +25 mV for activated (depolarized) heart areas are widely assumed. With this knowledge we propose the effective stopping criterion for iteration (11): as the solution range achieves 110 mV we break the algorithm and take curent iterate x_{k+1} as the optimal. With the information about solution range on each step we can also control rate of decay for the parameter λ making it closer to 1 when the desired threshold is almost achieved.

3. Results

To test the proposed algorithm we simulated three ectopic beats initiated in different heart locations. The Gaussian noise of 30 dB was added to the ECG signal.

For reconstruction we used a heart coarse mesh with the resolution of 4 mm (3444 points) whereas forward calculations were performed in a fine heart model with approximately 1mm average distance between nodes in finite element grid [2].

We compared our method with standard SVD based Tikhonov reconstruction with L-curve for the regularization parameter choice. The simulated data and reconstructed potentials for time instances 30ms and 60 ms are shown in the Fig. 1-2.

It's seen from the Fig. 1 that at the initial time instances of activation sequence when the ECG signal is weak the L-curve solution is underregularized. The proposed method in contrast identifies the ectopic foci more precisely.

For further time points the both algorithms show comparable performance. The correlation coefficients for the whole considered time period 20-100 ms are given in the Fig. 3-5.

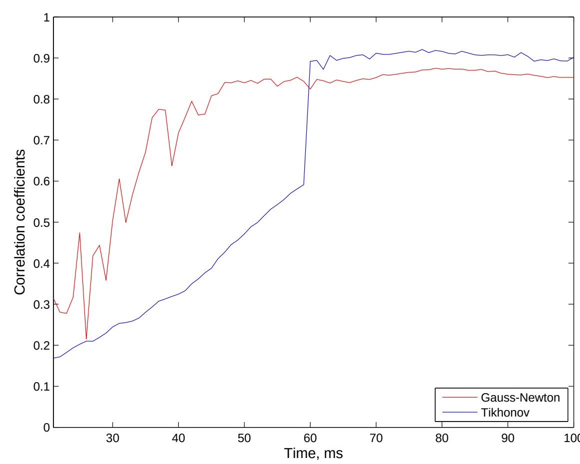

Figure 3. Correlation coefficients for the time period 20-100 ms, simulation1

4. Consclusions and discussion

In this paper we propose a new method for choosing the regularization parameter in the transmembrane potential based inverse problem of ECG. The made assumptions on the solution range are rather physiological than empirical and don't count on the model error, which makes the algorithm an attractive alternative to existing methods.

The potentials are continuous functions in time, and in the future work we will extend the algorithm to make the

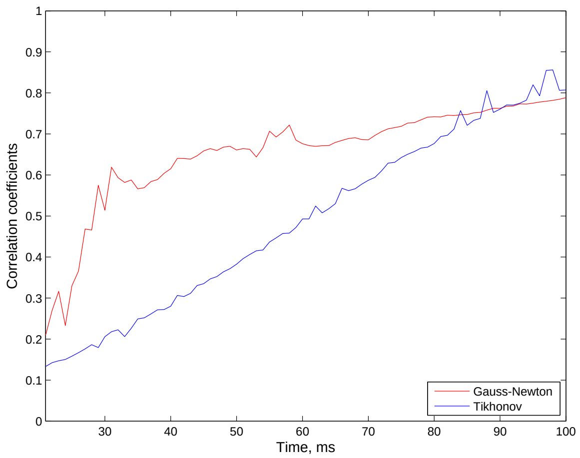

Figure 4. Correlation coefficients for the time period 20-100 ms, simulation2

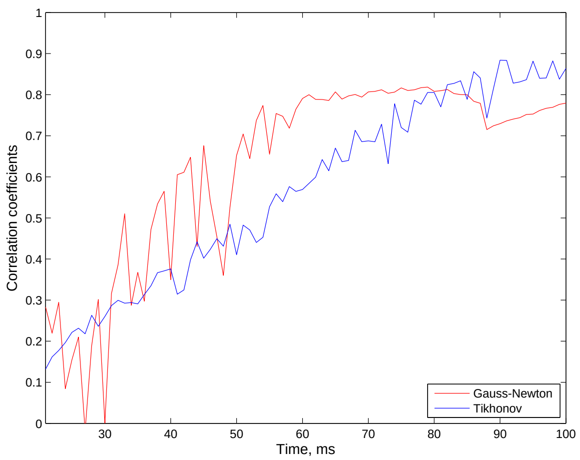

Figure 5. Correlation coefficients for the time period 20-100 ms, simulation3

parameter choice more automatical based on previous iterations in Gauss-Newton routine.

Acknowledgements

This work was funded by German Research Association under grant DO 637/13-1

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