Representation of the Cardiac Electrical Activity in the Form of a Double Layer Potential

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Abstract

In this paper, the representation of the potential of the cardiac electric field by the double layer potential is theoretically investigated.

1. Introduction

There is a hypothesis (see, for instance [1]) that the cardiac electrical potential can be represented as the potential of a double layer (defined on the myocardium surface) with a density proportional to the surface transmembrane potential. This hypothesis underlies the "solid angle" method for studying the ventricles electrical potentials [1]. The double layer approach can also be applied to solve the inverse problem of electrocardiography. In this paper, we consider this hypothesis based on the simplified version of the bidomain model.

2. Definitions and preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a relatively smooth boundary $\partial\Omega$. The double layer potential u for the Laplacian in $\mathbb{R}^3$ is defined as:

$$u = \frac{1}{4\pi} \int\limits_{\partial\Omega} \frac{\partial}{\partial n} \frac{1}{|x-y|} w(y) dS_y, x \in \mathbb{R}^3, y \in \mathbb{R}^3,$$

or

$$u = W_{\partial\Omega} w,$$

where n is the outer unit normal vector defined for almost all y on $\partial\Omega$, $\frac{\partial}{\partial n}\frac{1}{|x-y|}$ is the conormal derivative of the fundamental solution to the Laplace equation in $\mathbb{R}^3$, $W_{\partial\Omega}$ is the double layer operator, w is the double layer density.

We assume that $\partial\Omega$ is a Lipshitz boundary and $w \in L_2(\partial\Omega)$. This choice is motivated by the fact that the cardiac surface is usually approximated by a triangular surface for the numerical computations and the solid angle approach [1] considers the double layer density a piecewise constant function. The theory of elliptic boundary value problems for Lipshitz domains with boundary data in L_p is still far from complete. In this paper we will apply the non-tangential limit concept (see, for instance [2] and the literature cited there).

For points $x \in \partial\Omega$ we define an interior non-tangential approach region:

$$Q_{int}(x) = \{y \in \Omega : |x-y| \leq (1+\alpha) \cdot dist(y, \partial\Omega)\}$$

with fixed $\alpha > 0$. In the same way we define the exterior non-tangential approach region $Q_{ext}(x)$ for points $y \in \mathbb{R}^3 \setminus \overline{\Omega}$.

We define the interior and exterior restrictions of a function $u(x), x \in \mathbb{R}^3$ on $\partial\Omega$ by the non-tangential limits:

$$T^{int}_{\partial\Omega} u(x) = \lim_{p\to x} u(p), x \in \partial\Omega, p \in Q_{int},$$

$$T^{ext}_{\partial\Omega} u(x) = \lim_{p\to x} u(p), x \in \partial\Omega, p \in Q_{ext},$$

In the same manner we define the interior and exterior conormal derivatives of u on $\partial\Omega$:

$$\partial^{int}_{\partial\Omega} u(x) = \lim_{p\to x} \langle \nabla u(p), n\rangle, x \in \partial\Omega, p \in Q_{int},$$

$$\partial^{ext}_{\partial\Omega} u(x) = \lim_{p\to x} \langle \nabla u(p), n\rangle, x \in \partial\Omega, p \in Q_{ext}.$$

For a function u in Ω we define an interior non-tangential maximal function $u^{max}_{int}(x) = sup|u(y)|$, $x \in \partial\Omega$, $y \in Q_{int}$. Similarly, we define an exterior non-tangential maximal function $u^{max}_{ext}(x) = sup|u(y)|$, $x \in \partial\Omega, y \in Q_{ext}$ for a function u in $\mathbb{R}^3 \setminus \overline{\Omega}$.

We assume that the interior and exterior non-tangential maximal functions $(\nabla u)_{max}$ for gradient ∇u of u belongs to $L_2(\partial\Omega)$. Then $T^{int(ext)}_{\partial\Omega} u(x)$ and $\partial^{int(ext)}_{\partial\Omega} u(x)$ are unique, exist a.e. on $\partial\Omega$ and belong to $L_2(\partial\Omega)$. Moreover, the classical jump properties of the single layer and double layer potentials holds true. In particular,

$$T^{int}_{\partial\Omega} u(x) = (-\tfrac{1}{2} I + K) w \equiv \widetilde{W}^{int}_{\partial\Omega} w,$$

$$T^{ext}_{\partial\Omega} u(x) = (\tfrac{1}{2} I + K) w \equiv \widetilde{W}^{ext}_{\partial\Omega} w,$$

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where $Ku = p.v.\frac{1}{4\pi}\int\limits_{\partial\Omega}\frac{\langle(x-y),n\rangle}{|x-y|^3}w(y)dS_y$.

$T^{int}_{\partial\Omega}u(x) - T^{ext}_{\partial\Omega}u(x) = w; \partial^{int}_{\partial\Omega}u(x) - \partial^{ext}_{\partial\Omega}u(x) = 0.$

$\widetilde{W}^{int}_{\partial\Omega}c = c$, $\widetilde{W}^{ext}_{\partial\Omega}c = 0$, where c is a constant.

Operators $\widetilde{W}^{int}_{\partial\Omega}$ and $\widetilde{W}^{ext}_{\partial\Omega}$ acting from $L_2(\partial\Omega)$ to $L_2(\partial\Omega)$ are continuous, operator $\widetilde{W}^{int}_{\partial\Omega}$ is continuously invertible. Unfortunately, when $\partial\Omega$ is Lipshitz surface, $\widetilde{W}^{int}_{\partial\Omega}$ and $\widetilde{W}^{ext}_{\partial\Omega}$ are not necessary Fredholm's operators.

It is also easy to prove the converse of the double layer jump properties:

Proposition 2.1: Let U_{int} and U_{ext} be harmonic functions defined in Ω and $\mathbb{R}^3 \setminus \overline{\Omega}$ respectively, such that $\partial^{int}_{\partial\Omega}u = \partial^{ext}_{\partial\Omega}u$ on $\partial\Omega$ and $U_{ext} = \mathcal{O}(\frac{1}{|x|})$ as $|x| \to \infty$. Then there is a unique double layer density $w \in L_2(\partial\Omega)$, namely $w = T^{int}_{\partial\Omega}U_{int} - T^{ext}_{\partial\Omega}U_{ext}$, such that $U_{int} = W_{\partial\Omega}w, U_{int} = W_{\partial\Omega}(w + const)$.

The boundary value data in $L_2(\partial\Omega)$ for the calassical boundary value problems are understood in the sense of the non-tangential limits. Subject to $(\nabla u)_{max} \in L_2(\partial\Omega)$, the classical results on the solvability of the interior Dirichlet problem, and the interior and exterior Neumann problems for the Laplacian remain true. In particular, the solution to the Dirichlet problem is always represented by the potential of a double layer whose density is unique.

Let's take a closer look at the exterior Dirichlet problem (EDP):

$$
\begin{aligned}
&\Delta U_{ext} = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega},\\
&T^{ext}_{\partial\Omega}U_{ext} = f,\\
&U_{ext} = O(\frac{1}{|x|}) \text{ as } |x| \to \infty.
\end{aligned}
$$

The problem is uniquely solvable. Under the stronger "decay on infinity" condition: $u = O(\frac{1}{|x|^2})$ as $|x| \to \infty$ its solution is the potential of a double layer with density $w \in L_2(\partial\Omega)$. The double layer density w can be found by solving the operator equation: $\widetilde{W}^{ext}_{\partial\Omega}w = f$. Its solution is defined up to an arbitrary additive constant and does not exist for all $f \in L_2(\partial\Omega)$. The well-known Fredholm solvability criteria for this equation can not be applied. However, Proposition 2.1 shows that, as in the Fredholm case, the solution to EDP is the double layer potential if the conjucate interior Neumann problem:

$$
\begin{aligned}
&\Delta U_{int} = 0 \text{ in } \Omega,\\
&\partial^{int}_{\partial\Omega}U_{int} = \partial^{ext}_{\partial\Omega}U_{ext} \text{ on } \partial\Omega
\end{aligned}
$$

is solvable.

The solvability criterium for the interior Neumann problem allow us to formulate the following statements.

Proposition 2.2: Let U_{ext} be a solution to EDP. There is a double layer density $w \in L_2(\partial\Omega)$ such that U_{ext} is the double layer potential: $U_{ext} = W_{\partial\Omega}w$ if and only if $\int\limits_{\partial\Omega}\partial^{ext}_{\partial\Omega}U_{ext}(x)ds = 0$.

Corollary 2.3: Let $\Theta \subset \mathbb{R}^3$ be a bounded Lipshitz domain with boundary $\partial\Theta$ such that $\overline{\Omega} \subset \Theta$. Then the solvability condition of the Proposition 2.2 is equivalent to the condition: $\int\limits_{\partial\Omega}\partial^{ext}_{\partial\Theta}U_{ext}(x)ds = 0$.

3. Results

Let $\Omega \subset \mathbb{R}^3$ be the body domain with the boundary Γ_b and Ω_m: $\overline{\Omega}_m \subset \Omega$ be the myocardium domain (the atria or ventricles) with the boundary Γ_b. We call a domain $\Omega \setminus \Omega_m$ with the boundary $\Gamma_b \cup \Gamma_m$ as the extra-myocardial domain and denote it as Ω_b. We assume that Γ_b and Γ_m are Lipschitz boundaries.

The analysis presented in this paper is based on the bidomain model of electrical activity of the heart, which govers the intracellular (u_i), extracellular (u_e) and extra-myocardial (u_m) electrical potentials. We use a reduced and simplified version of the bidomain model. Namely, we consider only the elliptic equation of the bidomain model (see [3]) and assume the intracellular (σ_i), extracellular (σ_e) and extra-myocardial (σ_b) conductivities to be scalar real positive constants. If the torso is surrounded by a non-conductive medium (air), then the bidomain model can be formulated as follows.

Model 1.

$$
\begin{aligned}
&\sigma_i\Delta u_i + \sigma_e\Delta u_e = 0 \text{ in } \Omega_m,\\
&\Delta u_b = 0 \text{ in } \Omega_b\\
&T^{int}_{\Gamma_m}u_e = T^{ext}_{\Gamma_m}u_b \text{ on } \Gamma_m,\\
&\partial^{int}_{\Gamma_m}u_i = 0, \sigma_e\partial^{int}_{\Gamma_m}u_e = \sigma_b\partial^{ext}_{\Gamma_m}u_b \text{ on } \Gamma_m,\\
&\partial^{int}_{\Gamma_b}u_b = 0.
\end{aligned} \tag{1}
$$

We also assume that the torso is surrounded by a conductive medium with electrical conductivity equal to the torso one and the extra-myocardial potential u_b is defined in $\mathbb{R}^3 \setminus \overline{\Omega}_m$. This gives us the second version of the bidomain model:

Model 2.

$$
\begin{aligned}
&\sigma_i\Delta u_i + \sigma_e\Delta u_e = 0 \text{ in } \Omega_m,\\
&\Delta u_b = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega}_m,\\
&u_b = \mathcal{O}(\frac{1}{|x|}) \text{ as } |x| \to \infty,\\
&T^{int}_{\Gamma_m}u_e = T^{ext}_{\Gamma_m}u_b \text{ on } \Gamma_m,\\
&\partial^{int}_{\Gamma_m}u_i = 0, \sigma_e\partial^{int}_{\Gamma_m}u_e = \sigma_b\partial^{ext}_{\Gamma_m}u_b \text{ on } \Gamma_m,
\end{aligned} \tag{2}
$$

In addition, for both models we define the myocardium surface trasmenbrane potential v on Γ_m as

$v = T^{int}_{\Gamma_m} u_i - T^{int}_{\Gamma_m} u_e$.

For Models 1 and 2 we assume that the non-tangential maximal functions for gradients of $\sigma_i u_i + \sigma_e u_e$ and u_b belongs to $L_2(\Gamma_m)$ and $L_2(\Gamma_b)$ respectively.

Theorem 3.1: Let u_i, u_e, u_b satisfy Model 2. There is a double layer density $w \in L_2(\Gamma_m)$ such that u_b is the the double layer potential: $u_b = W_{\Gamma_m}(w + const)$. The double layer density w is proportional to the surface trnsmebrane potential v: $w = \frac{\sigma_i}{\sigma_i + \sigma_e} v$ if and only if $\sigma_b = \sigma_i + \sigma_e$.

Proof: We consider two functions: u_b defined in $\mathbb{R}^3 \setminus \overline{\Omega}_m$ and $\phi = \frac{\sigma_i u_i + \sigma_e u_e}{\sigma_b}$ defined in Ω_m. Functions u_b and ϕ are both harmonic. Acording to (2) $\partial^{int}_{\Gamma_m} \phi = \partial^{ext}_{\Gamma_m} u_b$ on Γ_m. Thus, according to Proposition 2.1 u_b is a restriction on $\mathbb{R}^3 \setminus \overline{\Omega}_m$ of the double layer potential with density w: $u_b = W_{\partial\Omega}(w + const)$ with $w = T^{int}_{\Gamma_m} \phi - T^{ext}_{\Gamma_m} u_b$.

Assume that $w = kv$ with a proportionality coefficient k. Using the definitions of ϕ and v and (2), that relation can be written as:

$T^{int}_{\Gamma_m}(\frac{\sigma_i u_i + \sigma_e u_e}{\sigma_b}) - T^{ext}_{\Gamma_m} u_e = k(T^{int}_{\Gamma_m} u_i - T^{int}_{\Gamma_m} u_e)$. It is easy to check that this equality is valid only if $\sigma_b = \sigma_i + \sigma_e$ and $k = \frac{\sigma_i}{\sigma_i + \sigma_e}$.

■

Theorem 3.2: Let u_b in Ω_b satisfy Model 1 and U in $\mathbb{R}^3$ be a potenial of a double layer defined on the myocardium boundary Γ_m. If $T^{int}_{\Gamma_b} u_b \neq 0$, there is no double layer potential U such that u_b is a restriction of U on Ω_b.

Proof: Assume that u_b is a restriction of the double layer potential U on Ω_b and $T^{int}_{\Gamma_b} u_b \neq 0$. According to (1) $\partial^{int}_{\Gamma_m} U = 0$. The double layer potential U is continuous when crossing the boundary Γ_b, hence $\partial^{ext}_{\Gamma_b} U = \partial^{int}_{\Gamma_b} U = 0$. Let U_{out} be a restriction of U on $\mathbb{R}^3 \setminus \overline{\Omega}$. Function U_{out} is a solution to the exterior Neumann problem:

$$\Delta U_{out} = 0 \text{ in } \mathbb{R}^3 \setminus \overline{\Omega},$$

$$\partial^{ext}_{\Gamma_b} U_{out} = 0 \text{ in } \Gamma_b,$$

$$U_{out} = \mathcal{O}(\frac{1}{|x|}) \text{ in } |x| \to \infty.$$

Obviously, $U_{out} = 0$. Thus, $T^{ext}_{\Gamma_b} U_{out} = 0$. Double layer potential U is continuous when crossing the boundary Γ_b, hence $T^{int}_{\Gamma_b} U = T^{int}_{\Gamma_b} u_b = 0$ on Γ_b. This fact contradicts the assumption that $T^{int}_{\Gamma_b} u_b \neq 0$.

■

Although the body electrical potential u_b satisfying Model 1 cannot be presented as a double layer potential we can state a problem to find a double layer potential U which "coincides" with u_b only on Γ_m.

Problem 1. Let u_b be a function satisfying Model 1. It is required to find a density $w \in L_2(\Gamma_m)$ of the double layer potential U such that $T^{ext}_{\Gamma_m} U = T^{ext}_{\Gamma_m} u_b = \widetilde{W}^{ext}_{\Gamma_m} w$ on Γ_m: **(a)** if $T^{ext}_{\Gamma_m} u_b$ is given: $T^{ext}_{\Gamma_m} u_b = f_m \in L_2(\Gamma_m)$; **(b)** if $T^{int}_{\Gamma_b} u_b$ is given: $T^{ext}_{\Gamma_b} u_b = f_b \in L_2(\Gamma_b)$.

Problem 1(b) can be called the inverse electrocadiography problem in terms of the density of the double layer. Note, that even if $\sigma_b = \sigma_i + \sigma_e$ the density w of such double layer potential U is not nesessary proportional to the transmembrane potential v.

To formulate the solvability results to Problem 1, we introduce two operators. Let U be a solution to EDP in $\mathbb{R}^3 \setminus \overline{\Omega}_m$ and $f \in L_2(\Gamma_m)$ be the EDP boundary condidtion. The first operator D is a linear continous map of the boundary datum f of EDP to the interior non-tangential limit of its solution on Γ_b: D: $f \in L_2(\Gamma_m) \to T^{int}_{\Gamma_b} U \in L_2(\Gamma_b)$. The second operator is the continuous linear Dirichlet-to-Neumann map N_{Γ_m} on Γ_m: N_{Γ_m}: $f \in L_2(\Gamma_m) \to \partial^{ext}_{\Gamma_m} U \in L_2(\Gamma_m)$. In the same way, we define the Dirichlet-to-Neumann map N_{Γ_b} on Γ_b: N_{Γ_b}: $T^{ext}_{\Gamma_b} U \in L_2(\Gamma_b) \to \partial^{ext}_{\Gamma_b} U \in L_2(\Gamma_b)$. Note that operators D and N_{Γ_m}, N_{Γ_b} can be expressed explicitly by the single layer and double layer potential operators in different ways.

We will need the following result.

Let U be a solution of EDP in $\mathbb{R}^3 \setminus \overline{\Omega}_m$ with the boundary datum $f = T^{ext}_{\Gamma_m} u_b$. Then

$$T^{int}_{\Gamma_b} U = T^{ext}_{\Gamma_b} U = (\widetilde{W}^{ext}_{\Gamma_b} - D \cdot T^{int}_{\Gamma_b} W_{\Gamma_b}) \cdot T^{int}_{\Gamma_b} u_b. \quad (3)$$

This statement is a special case (see [4], Section 4) of a much more general result obtained in [5].

Theorem 3.3: Problem 1(a) is solvable if and only if $\int_{\Gamma_m} N_{\Gamma_m} f_m ds = 0$. Problem 1(b) is solvable if and only if $\int_{\Gamma_b} N_{\Gamma_b} B f_b ds = 0$, where $B \equiv \widetilde{W}^{ext}_{\Gamma_b} - D \cdot T^{int}_{\Gamma_b} W_{\Gamma_b}$. The solutions to Problem 1 (a) and Problem 1 (b) if exist are unique up to an arbitrary additive constant.

Proof: The statements of the theorem follows directly from Proposition 2.2, Corollary 2.3 and formula (3).

■

4. Discussion and conclusions

The state-of-art potential theory allow us to consider the potential of a double layer with square integrable (including piecewise constant) density given on a Lipschitz surface approximating the boundary of the myocardium.

If the body is surrounded by an infinite electrically conductive medium with a conductivity equal to that of the body (Model 2), then the electrical potential in the extramyocardial domain is always represented by the potential of a double layer defined on the myocardium surface. The density of a double layer representing a given extramyocardial potential is unique up to an arbitrary additive constant. If the body conductivity is equal to the sum of the intracellular and extracellular conductivities, then the double layer density is proportional to the transmembrane potential on the myocardium surface.

Under the assumptions that the medium surrounding the body does not significantly affect the electrophysiological processes of the myocardium and the above relation of the electrical conductivities is close to physiological, the "solid angle" method [1] can be applied to analyse the electrical field of the heart.

If the body is surrounded by air (Model 1), then the nonzero cardiac electrical potential in the extra-myocardial domain cannot be represented by the potential of a double layer defined on the myocardium surface. The cardiac electrical potential of the myocardium surface is presented by the double layer potential only subject to the conditions obtained in this work. Moreover, the double layer density is not necessarily proportional to the transmembrane potential on the surface of the myocardium, even if electrical conductivity relations (3) are met. Therefore, the representation of the solution to the inverse problem of electrocardiography in terms of a double layer has significant limitations.

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